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QUANTUM GROUPS AND NON-COMMUTATIVE COMPLEX ANALYSIS

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Non-commutative analogues of function algebras on the unit ball are considered in the context of quantum group theory. The maximum principle for holomorphic functions in the quantum ball is expounded. An invariant integral on the quantum analogue of function algebra on sphere is used to produce a faithful *-representation of the latter C*-algebra. The Shilov boundary for the quantum analogue of the algebra of holomorphic functions is described.
KEYWORDS: quantum unit ball, C*-algebra, completely isometric map, invariant integral, the Shilov boundary.

The problem of uniform approximation by analytic polynomials on a compact $\mathrm{K} \subset \mathbb{C}$ made an essential impact to the function theory and the theory of commutative Banach algebras. This problem was solved by S. Mergelyan. Much later, a theory of uniform algebras was developed and an abstract proof of Mergelyan's theorem was obtained.

The work by W. Arveson [1] starts an investigation of non-commutative analogs for uniform algebras. In particular, a notion of the Shilov boundary for a subalgebra of a C^*-algebra have been introduced therein. So, the initial results of non-commutative complex analysis were obtained. We assume the basic concepts of that work known to the reader.

In mid'90-s an investigation of quantum analogs for bounded symmetric domains has been started within the framework of the quantum group theory [2]. The simplest of those is a unit ball in $\mathbb{C}^n$. Our goal is to explain that the quantum sphere is the Shilov boundary for this quantum domain. The subsequent results of the authors on non-commutative function theory and quantum groups are available at www.arxiv.org . Specifically, we obtained some results on weighted Bergman spaces, the Berezin transform, and the Cauchy-Szegö kernels for quantum bounded symmetric domains introduced in [3].

In what follows, the complex numbers are assumed as a ground field and all the algebras are assumed unital. In what follows $q \in (0,1)$.

THE QUANTUM BALL

To introduce a quantum unit ball, consider a *-algebra $Pol(\mathbb{C}^n)_q$ given by the generators $z_1, z_2, ..., z_n$ the defining relations $z_j z_k = q z_k z_j$ for $j < k$,

$$z_j^* z_k = q z_k z_j^*, \qquad j \neq k, \qquad z_j^* z_j = q^2 z_j z_j^* + (1-q^2)\left(1 - \sum_{k>j} z_k z_k^*\right). \tag{1}$$

This *-algebra has been introduced by W. Pusz and S. Woronowicz [4] where one can find a description (up to unitary equivalence) of its irreducible *-representations T. One can demonstrate that $0 < \| f \| \stackrel{def}{=} \sup_T \| T(f) \| < \infty$ for all non-zero $f \in Pol(\mathbb{C}^n)_q$ and that its C^*-enveloping algebra $C(\overline{\mathbb{D}})_q$ is a q-analogue for the C^*-algebra of continuous functions in the closed unit ball in $\mathbb{C}^n$. A plausible description of this C^*-algebra has been obtained by D. Proskurin and Yu. Samoilenko [5].

To introduce a quantum unit sphere, consider a closed two-sided ideal J of the C^*-algebra $C(\overline{\mathbb{D}})_q$ generated by $1 - \sum_{j=1}^{n} z_j z_j^*$. Obviously, the C^*-algebra $C(\partial\mathbb{D})_q \stackrel{def}{=} C(\overline{\mathbb{D}})_q / J$ is a q-analogue for the algebra we need. Thus the canonical onto morphism

$$j_q : C(\overline{\mathbb{D}})_q \to C(\partial\mathbb{D})_q \tag{2}$$

is a q-analogue for the restriction operator of a continuous function onto the boundary of the ball.

The closed subalgebra $A(\overline{\mathbb{D}})_q \subset C(\partial\mathbb{D})_q$ generated by $z_1, z_2, ..., z_n$ is a q-analogue for the algebra of continuous functions in the closed ball, which are holomorphic in its interior.

Let $j_{A(\overline{\mathbb{D}})_q}$ be the restriction of the homomorphism j_q onto the subalgebra $A(\overline{\mathbb{D}})_q$.

Theorem. *The homomorphism $j_{A(\overline{\mathbb{D}})_q}$ is completely isometric.*[1]

Here is a sketch of a well known trick that we use to prove the Theorem. As above, let A be a subalgebra of a C*-algebra B, J a closed two-sided *-ideal of this C*-algebra, $j: B \to B/J$ the canonical homomorphism, and $j_A : A \to B/J$ its restriction onto A. We need to prove a complete isometricity for j_A. We fix a faithful *-representation T of B in a Hilbert space $\mathcal{H}$ and construct a dilation of its restriction T_A onto A. More precisely, we construct such *-representation $\tilde{T}$ of B/J in a Hilbert space

$$\tilde{\mathcal{H}} \supset \mathcal{H} \tag{3}$$

that

$$T(a) = P_{\mathcal{H}} \cdot \tilde{T}(j(a))|_{\mathcal{H}} \tag{4}$$

for all $a \in A$, with $P_{\mathcal{H}}$ being the orthogonal projection onto the subspace $\mathcal{H}$. It follows from the inequalities

$$\| a \| \geq \| j(a) \|, \quad \| a \| = \| T(a) \| \leq \| \tilde{T}(j(a)) \| \leq \| j(a) \|, \quad a \in A, \tag{5}$$

that the operator j_A is isometric. Replace here $\mathcal{H}$ and $\tilde{\mathcal{H}}$ by $\mathcal{H} \otimes \mathbb{C}^k$ and $\tilde{\mathcal{H}} \otimes \mathbb{C}^k$ respectively to obtain a proof that j_A is in fact completely isometric.

Note that, instead of the embedding (3), one can use in the definition of the dilation the isometry $i : \mathcal{H} \supset \tilde{\mathcal{H}}$ by replacing $P_{\mathcal{H}}$ with the operator i^*, and by replacing (4) with the assumption

$$T(a) = i^* \cdot \tilde{T}(j(a)) \cdot i, \quad a \in A. \tag{6}$$

This result is a q-analogue of the well-known maximum principle for holomorphic functions. By an Arveson's definition [1] this means that J is a boundary ideal for the subalgebra $A(\overline{\mathbb{D}})_q$. A proof of this theorem elaborates the methods of quantum group theory and theory of unitary dilations [6].

AN INVARIANT INTEGRAL ON THE QUANTUM SPHERE

While producing dilation, we use essentially a q-analogue of the natural representation of the algebra $C(\partial\mathbb{D})$ in $L^2(\partial\mathbb{D})$. Here is a description of this q-analogue.

Let $\mathbb{C}[Mat_n]_q$ be the algebra of holomorphic polynomials on the quantum space of $n \times n$-matrices introduced in [2]. It is an algebra with generators $\{t_{i,j}\}_{i,j=1,2,\ldots,n}$ and the defining relations

$$\begin{aligned}
t_{i_1,j} \cdot t_{i_2,j} &= q t_{i_2,j} \cdot t_{i_1,j}, && i_1 < i_2; \\
t_{i,j_1} \cdot t_{i,j_2} &= q t_{i,j_2} \cdot t_{i,j_1}, && j_1 < j_2; \\
t_{i_1,j_1} \cdot t_{i_2,j_2} &= t_{i_2,j_2} \cdot t_{i_1,j_1}, && i_1 < i_2 \ \&\ j_1 > j_2; \\
t_{i_1,j_1} \cdot t_{i_2,j_2} &= t_{i_2,j_2} \cdot t_{i_1,j_1} + (q - q^{-1}) t_{i_1,j_2} \cdot t_{i_2,j_1}, && i_1 < i_2 \ \&\ j_1 < j_2.
\end{aligned} \tag{7}$$

Introduce the notation $\mathbf{t}$ for the matrix $(t_{i,j})$. Recall that the center of $\mathbb{C}[Mat_n]_q$ is generated by the well known quantum determinant $\det_q \mathbf{t}$.

Let us consider the Hopf algebra $\mathbb{C}[SL_n]_q = \mathbb{C}[Mat_n]_q / (\det_q \mathbf{t} - 1)$ with coproduct, counit, and antipode being given by

$$\begin{gathered}
\Delta : t_{i,j} \mapsto \sum_{k=1}^{n} t_{i,k} \cdot t_{k,j}, \qquad \varepsilon : t_{i,j} \mapsto \delta_{i,j}, \\
S : t_{i,j} \mapsto (-q)^{i+j-2n} (\det_q \mathbf{t})^{-1} \cdot \det_q \mathbf{t}_{i,j}, \quad i,j = 1,2,\ldots,n,
\end{gathered} \tag{8}$$

with $\mathbf{t}_{i,j}$ being the natrix derived from $\mathbf{t}$ via descarding line i and column j. Equip $\mathbb{C}[SL_n]_q$ with an involution

$$t_{i,j}^* = S(t_{i,j}), \quad i,j = 1,2,\ldots,n, \tag{9}$$

[1] A definition of completely isometric linear map between subalgebras of C*-algebras can be found in [12].

to get a Hopf-*-algebra $\mathbb{C}[SU_n]_q$, which is a q-analogue of the algebra $\mathbb{C}[SU_n]$ of regular functions on the group SU_n. A description of all irreducible *-representations of $\mathbb{C}[SU_n]_q$ up to unitary equivalence is due to L.I. Korogodski and Ya. Soibelman [7]. It follows from the definitions that for all $f \in \mathbb{C}[SU_n]_q$, one has a finite supremum $\| f \| = \sup \| \mathcal{T} f \|$, with $\mathcal{T}$ varies over the set of unitary equivalence classes of irreducible *-representations of $\mathbb{C}[SU_n]_q$. A completion of this algebra with respect to the norm $\| f \|$ is a q-analogue of the C*-algebra $C[SU_n]$ of continuous functions on the group SU_n and is denoted by $C[SU_n]_q$.

The unit sphere $\partial\mathbb{D}$ is a homogeneous space for the group SU_n, which allows one, given a point $g \in \partial\mathbb{D}$, to produce an embedding $C(\partial\mathbb{D}) \to C[SU_n]$, $f(p) \mapsto f(g^{-1}p)$, $f \in C(\partial\mathbb{D})$. A similar embedding is also present in the quantum case. As a direct consequnce of the definitions of the C*-algebras $C(\partial\mathbb{D})_q$, $C[SU_n]_q$ one has

Proposition 1. *The map $z_a \mapsto z_a^n$, $a = 1,2,\dots,n$, is uniquely extendable up to an injective homomorphism of C*-algebras $C(\partial\mathbb{D})_q \to C[SU_n]_q$.*

Recall [8] that a state on a unital C*-algebra is a positive functional ν with $\nu(1) = 1$. In the classical case (q=1) the integral with respect to the Haar measure is an invariant state on $C[SU_n]$, and $\int_{SU_n} \| f \|^2 d\nu > 0$ for $f \neq 0$. S. Woronowicz [9] proved the existence of an invariant (in the context of quantum group theory) state

$$\nu : C(SU_n)_q \to \mathbb{C}, \qquad f \mapsto \int_{(SU_n)_q} f d\nu \tag{10}$$

on the quantum SU_n. It is known that $\int_{(SU_n)_q} \| f \|^2 d\nu > 0$ for $f \neq 0$. An explicit form of an invariant integral ν was found by N. Reshetikhin and M. Yakimov [10]. The embedding $i : C(\partial\mathbb{D})_q \to C[SU_n]_q$ allows one to transfer the state ν from $C[SU_n]_q$ onto $C(\partial\mathbb{D})_q$:

$$\int_{(\partial\mathbb{D})_q} f d\nu \overset{def}{=} \int_{(SU_n)_q} i(f) d\nu \tag{11}$$

The state $\nu : C(\partial\mathbb{D})_q \to \mathbb{C}$ defined this way is a q-analogue of the SU_n-invariant integral $\nu : C(\partial\mathbb{D}) \to \mathbb{C}$. We use this state ν to get a faithful representation of the C*-algebra $C(\partial\mathbb{D})_q$ via the Gelfand-Naimark-Sigal construction. A scalar product equips $C(\partial\mathbb{D})_q$ with a structure of pre-Hilbert space. Its completion $L^2(\partial\mathbb{D})_q$ is a q-analogue for the space of square summable functions on the sphere $\partial\mathbb{D}$. It follows from the definition that $C(\partial\mathbb{D})_q \to L^2(\partial\mathbb{D})_q$ is an embedding. So the Gelfand-Naimark-Sigal construction associates to ν a representation R of the above C*-algebra $C(\partial\mathbb{D})_q$ in $L^2(\partial\mathbb{D})_q$ such that $R(f)\psi = f\psi$ for all $f, \psi \in C(\partial\mathbb{D})_q$.

THE SHILOV BOUNDARY OF THE UNIT BALL

It follows from Theorem that the two-sided ideal $J \subset C(\overline{\mathbb{D}})_q$ generated by $1 - \sum_{j=1}^{n} z_j z_j^*$ is a boundary ideal for the subalgebra $A(\overline{\mathbb{D}})_q$. Due to the deep result of M. Hamana [11] (on existence of the Shilov boundary) there exists the largest among the boundary ideals for the subalgebra $A(\overline{\mathbb{D}})_q$. After the work of Arveson [1], it is called the Shilov boundary ideal for $A(\overline{\mathbb{D}})_q$. The quotient algebra associated to this ideal is a non-commutative analogue for the function algebra on the Shilov boundary of quantum ball.

Proposition 2. *J is the Shilov boundary ideal for $A(\overline{\mathbb{D}})_q$.*

Proof. Consider the C*-algebra $C(\partial\mathbb{D})_q$ together with its unital subalgebra $A = j_q \cdot A(\mathbb{D})_q$. The target statement is now due to the Theorem by Hamana mentioned above and the following result.

Proposition 3. *The only boundary ideal of the C*-algebra $C(\partial\mathbb{D})_q$ for A is the zero ideal.*

CONCLUSIONS

This paper considers non-commutative analogues of function algebras on the unit ball in the context of quantum group theory, along with suitable embeddings of the associated quantum analogues of function algebras on the sphere. An analogue of the maximum principle for holomorphic functions in this setting is formulated in terms of completely isometric maps. An invariant integral on the quantum analogue of function algebra on sphere can be used to produce a faithful *-representation of the latter C*-algebra. The Shilov boundary for the quantum analogue of the algebra of holomorphic functions is described. A detailed exposition of proofs is available in [12].

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КВАНТОВЫЕ ГРУППЫ И НЕКОММУТАТИВНЫЙ КОМПЛЕКСНЫЙ АНАЛИЗ

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В рамках теории квантовых групп рассматриваются некоммутативные аналоги алгебр функций на единичном шаре. Излагается принцип максимума для голоморфных функций в квантовом шаре. С помощью инвариантного интеграла на квантовом аналоге алгебр функций на сфере строится точное представление этой C*-алгебры. Описана граница Шилова для квантового аналога алгебры голоморфных функций.

КЛЮЧЕВЫЕ СЛОВА: квантовый единичный шар, C*-алгебра, вполне изометрическое отображение, инвариантный интеграл, граница Шилова.