PACS:74.50.+r,74.80.Fp

INTEGRABLE STRING MODELS OF HYDRODYNAMICAL TYPE

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Received March 28, 2005.

The closed string model of hydrodynamical type is considered as a bi-Hamiltonian system. Nonlocal Poisson brackets were used to construct nonlinear integrable string models, which are not models of sigma-model type. The recursion operator was obtained. Second string Hamiltonian was obtained.
KEY WORDS: string model, integrable model, nonlocal brackets, hydrodynamical type model.

String model is a remarkable extension of General Relativity. One of the important problems is to study the set of exact classical solutions to string equations of motion. The sigma-model approach in string theory was prompted by the covariant path integral representation for the string theory scattering amplitudes and by studies of sigma-model as geometrical 2-dimensional quantum field theories. An independent development was the realization that a necessary condition on a string solution is that the sigma-model describing string propagation in background fields be conformal invariant.

There are several approaches that were used to construct exact string solutions: 1) start with a particular leading order solution of string parameter α' and show that α'-corrections are absent due to some special properties of background fields ("plane wave" type backgrounds, certain backgrounds with a covariantly constant Killing vector); 2) start with a known lagrangian Conformal Field Theory and represent it as a conformal sigma-model (gauged WZW models); 3) start with a leading order sigma-model path integral and prove that there exists such sigma-model which is conformal to all loop orders (F-model, chiral null model, magnetic flux tube model).This may be relevant for understanding the implications of string theory for cosmology and black holes.

Let us consider bi-Hamiltonian approach to the integrable string models.

The bi-Hamiltonian approach to the integrable systems was initiated by Magri [1] and was generalized by Das, Okubo [2]. Two Poisson brackets (PB) $\{u^a(\sigma),u^b(\sigma')\}_1 = P_1^{ab}(\sigma,\sigma')(u)$ and $\{u^a(\sigma),u^b(\sigma')\}_2 = P_2^{ab}(\sigma,\sigma')(u)$ are called compatible if any linear combination of these PB $c_1\{*,*\}_1 + c_2\{*,*\}_2$ is PB also. The functions $u^a(\tau,\sigma), a=0,1...D-1$ are local coordinates on a certain given smooth D-dimensional manifold M. The Hamiltonian operators $P_1^{ab}(\sigma,\sigma')(u), P_2^{ab}(\sigma,\sigma')(u)$ are the functions of local coordinates $u^a(\sigma)$. It is possible to construct recursion operators

$$R_c^a(\sigma,\sigma')(u) = \int_0^\pi P_2^{ab}(\sigma-\sigma'')P_{bc1}(\sigma''-\sigma')d\sigma'', (R(\sigma,\sigma')(u))^{-1}{}_c^a = \int_0^\pi P_1^{ab}(\sigma-\sigma'')P_{bc2}(\sigma''-\sigma')d\sigma''.$$

Any of degrees of recursion operator is Hamiltonian operator of new PB. It is possible to find two Hamiltonians H_1 and H_2 which are satisfy to bi-Hamiltonian conditions:

$$\frac{du^a(\sigma)}{d\tau} = \{u^a(\sigma),H_1\}_1 = \{u^a(\sigma),H_2\}_2, \frac{du^a(\sigma)}{d\tau_1} = \{u^a(\sigma),H_2\}_1 = \{u^a(\sigma),H_1\}_2. \quad (1)$$

The local PB of hydrodynamical type was introduced by Dubrovin, Novikov [3] for Hamiltonian description of the equations of hydrodynamics. The nonlocal PB hydrodynamical type was introduced by Ferapontov [4] and Mokhov, Ferapontov [5]. It was shown that to construct family of integrable systems enough to consider pencil from local PB and nonlocal PB. Integrable systems of hydrodynamical type are described by Hamiltonians of hydrodynamical type, which are not depend of derivatives of local coordinates. Integrable bi-Hamiltonian systems of hydrodynamical type was considered by Maltsev [6], Ferapontov [7], Mokhov [8,9], Pavlov [10], Novikov, Maltsev [11].

STRING MODEL

The string model in the constant background gravity field is described by the system of equations:

$$\ddot{x}^a - x''^a + \Gamma_{bc}^a(x)(\dot{x}^b\dot{x}^c - x'^b x'^c) = 0, \eta_{ab}(\dot{x}^a\dot{x}^b + x'^a x'^b) = 0, \eta_{ab}\dot{x}^a x'^b = 0,$$

where $\dot{x}^a=\frac{dx^a}{d\tau}, x'^a=\frac{dx^a}{d\sigma}, a=0,1...D-1$ and $x^a(\tau,\sigma)$ are local target space coordinates of world sheet coordinates τ,σ . The constant metric tensor η_{ab} is symmetric tensor. The closed string model is described by first kind constraints in the Hamiltonian formalism:

$$h_1=\frac{1}{2}(\eta^{ab}p_ap_b+\eta_{ab}x'^ax'^b)\approx 0, h_2=p_ax'^a\approx 0, \tag{2}$$

where $x^a(\tau,\sigma), p_a(\tau,\sigma)$ are the periodical functions on σ with the period on π .

The original PB are the canonical PB:

$$\{x^a(\sigma),p_b(\sigma')\}_1=\delta^a_b\delta(\sigma-\sigma'),\{x^a(\sigma),x^b(\sigma')\}_1=\{p_a(\sigma),p_b(\sigma')\}_1=0. \tag{3}$$

Let us introduce the local coordinates $u^a(\sigma),v^a(\sigma)$:

$$u^a=\frac{1}{\sqrt{2}}(\eta^{ab}p_b+x'^a),v^a=\frac{1}{\sqrt{2}}(\eta^{ab}p_b-x'^a) \tag{4}$$

First kind constraints have following form:

$$h_1=\frac{1}{2}\eta_{ab}(u^au^b+v^av^b),h_2=\frac{1}{2}\eta_{ab}(u^au^b-v^av^b)$$

and canonical PB are:

$$\{u^a(\sigma),u^b(\sigma')\}_1=P_1^{ab}(\sigma-\sigma')=\eta^{ab}\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma'),$$
$$\{v^a(\sigma),v^b(\sigma')\}_1=-\eta^{ab}\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma'),\{u^a(\sigma),v^b(\sigma')\}_1=0. \tag{5}$$

The Hamiltonian equations of motion under Hamiltonian $H_1=\int_0^\pi h_1(u(\sigma))d\sigma$ are describe two independent left and right movers: $\dot{u}^a=u'^a,\dot{v}^a=-v'^a,u^a(\tau,\sigma)=u^a(\tau+\sigma),v^a(\tau,\sigma)=v^a(\tau-\sigma).$ We will consider a motion of string, which is described by local coordinates $u^a(\tau,\sigma)$.

NONLOCAL POISSON BRACKETS

Local PB Dubrovin, Novikov in the flat coordinates is coincide with PB (5). Ferapontov [4] construct nonlocal PB following form:

$$\{u^a(\sigma),u^b(\sigma')\}_F=g^{ab}(u(\sigma))\frac{d}{d\sigma}\delta(\sigma-\sigma')-g^{ac}(u)\Gamma^b_{ck}(u)u'^k(\sigma)\delta(\sigma-\sigma')+$$
$$\sum_{\alpha=1}^{L}(w_\alpha(u(\sigma)))^a_ku'^k(\sigma)\nu(\sigma'-\sigma)(w_\alpha(u(\sigma')))^b_lu'^l(\sigma'), \tag{6}$$

where $\det(g^{ab}(u))\neq 0,$ the coefficients $g^{ab}(u),\Gamma^a_{cb}(u),(w_\alpha(u))^a_b,a,b,c=0,1...D-1;\alpha=1,2...L$ are the smooth functions of local coordinates. The bracket (6) is skew-symmetric if and only if:

1. $g^{ab}(u)=g^{ba}(u)$ is a symmetric contravariant metric,
2. $\Gamma^a_{bc}(u)$ is the Riemannian connection generated by metric $g^{ab}(u)$.

The bracket (6) satisfies Jacobi identity if and only if:

3. $\Gamma^a_{bc}(u)=\Gamma^a_{cb}(u)$ is symmetric connection, and metric $g^{ab}(u)$ and tensors $(w_\alpha(u))^a_b$ satisfy the Gauss-Codazzi equations for the submanifolds M^D with flat normal bundle embedded in pseudo-Euclidian space E^{D+L}:
4. $g^{ab}(u)(w_\alpha(u))^c_b=g^{cb}(u)(w_\alpha(u))^a_b$,
5. $\nabla_a(w_\alpha(u))^b_c=\nabla_c(w_\alpha(u))^b_a$, where ∇_a is the operator of covariant differentiation generated connection $\Gamma^a_{bc}(u)$.

6. $R^{ab}_{cd}(u)=\sum_{\alpha=1}^{L}[(w_\alpha(u))^a_d(w_\alpha(u))^b_c-(w_\alpha(u))^b_d(w_\alpha(u))^a_c$, where $R^{ab}_{cd}(u)=g^{ak}(u)R^b_{kcd}(u)$ is the Riemannian curvature tensor of metric $g^{ab}(u)$.

7. The tensors $(w_\alpha(u))^a_b$ are satisfy the Ricci equations for embedded surface:
$(w_\alpha(u))^a_b(w_\beta(u))^b_c=(w_\beta(u))^a_b(w_\alpha(u))^b_c$.

The PB (6) exactly corresponds to an D-mensional surface with flat normal bundle embedded in a pseudo-Euclidean space E^{D+L}. The covariant tensor $g_{ab}(u)$ is the first fundamental form. The tensors $(w_\alpha(u))^a_b$ are the corresponding Weingarten operators of this embedded surface and tensors $g_{ab}(u)(w_\alpha(u))^b_c$ are the corresponding second fundamental forms.

COMPATIBLE PAIRS OF POISSON BRACKETS

As was shown in [6-9] that Ferapontov nonlocal PB (6) compatible with constant local Dubrovin,Novikov PB (5) under Magri condition [1] if and only if it has following form:

$$\begin{aligned}\{u^a(\sigma),u^b(\sigma')\}_2 &= P_2^{ab}(\sigma,\sigma')(u)=[\eta^{ac}\frac{\partial F^b(u(\sigma))}{\partial u^c}+\eta^{bc}\frac{\partial F^a(u(\sigma))}{\partial u^c}-\\ &\eta^{ak}\eta^{bl}\sum_{\alpha=1}^{L}\frac{\partial\psi^\alpha(u(\sigma))}{\partial u^k}\frac{\partial\psi^\alpha(u(\sigma))}{\partial u^l}]\frac{d}{d\sigma}\delta(\sigma-\sigma')+[\eta^{ac}\frac{\partial^2F^b(u(\sigma))}{\partial u^c\partial u^k}-\\ &\eta^{ac}\eta^{bl}\sum_{\alpha=1}^{L}\frac{\partial^2\psi^\alpha(u(\sigma))}{\partial u^k\partial u^c}\frac{\partial\psi^\alpha(u(\sigma))}{\partial u^l}]u'^k\delta(\sigma-\sigma')+\\ &\eta^{ac}\eta^{bk}\sum_{\alpha=1}^{L}\frac{\partial^2\psi^\alpha(u(\sigma))}{\partial u^c\partial u^l}u'^l(\sigma)\nu(\sigma'-\sigma)\frac{\partial^2\psi^\alpha(u(\sigma'))}{\partial u^k\partial u^n}u'^n(\sigma'),\end{aligned} \tag{7}$$

where functions $F^a(u),\psi^\alpha(u),a=0,1...D-1,\alpha=1,2...L$ are the smooth functions of local coordinates.

The compatible PB (7) is PB if and only if the following equations are satisfied the equations:

$$\begin{aligned}&\frac{\partial^2F^a}{\partial u^k\partial u^l}\eta^{\ln}\frac{\partial^2F^b}{\partial u^n\partial u^c}=\frac{\partial^2F^b}{\partial u^k\partial u^l}\eta^{\ln}\frac{\partial^2F^a}{\partial u^n\partial u^c},\frac{\partial^2\psi^\alpha}{\partial u^k\partial u^l}\eta^{\ln}\frac{\partial^2\psi^\beta}{\partial u^n\partial u^c}=\frac{\partial^2\psi^\beta}{\partial u^k\partial u^l}\eta^{\ln}\frac{\partial^2\psi^\alpha}{\partial u^n\partial u^c},\\ &g^{ak}{}_{can}\eta^{bl}\frac{\partial^2F^a}{\partial u^k\partial u^l}=g^{bk}{}_{can}\eta^{al}\frac{\partial^2F^a}{\partial u^k\partial u^l},g^{ak}{}_{can}\eta^{bl}\frac{\partial^2\psi^\alpha}{\partial u^k\partial u^l}=g^{bk}{}_{can}\eta^{al}\frac{\partial^2\psi^\alpha}{\partial u^k\partial u^l},\\ &g^{ab}{}_{can}=\eta^{ac}\frac{\partial F^b}{\partial u^c}+\eta^{bc}\frac{\partial F^a}{\partial u^c}-\eta^{ak}\eta^{bl}\sum_{\alpha=1}^{L}\frac{\partial\psi^\alpha}{\partial u^k}\frac{\partial\psi^\alpha}{\partial u^l}.\end{aligned} \tag{8}$$

INTEGRABLE BI-HAMILTONIAN STRING MODELS OF HYDRODYNAMICAL TYPE

Consider the recursion operator generated by compatible PB (5) and PB (7):

$$R^a_b(\sigma,\sigma')(u)=\int_0^\pi P_2^{ac}(\sigma,\sigma'')(u)P_{1cb}(\sigma'',\sigma')(u)d\sigma'', \tag{9}$$

where $P_{1ab}(\sigma,\sigma')\equiv(P_1^{ab}(\sigma,\sigma'))^{-1}=\eta_{ab}\nu(\sigma-\sigma')$.

Let us apply recursion operator (9) to the closed string model in the constant background gravity field. Hamiltonian equations of motion with the Hamiltonian

$$H_1=\int_0^\pi\eta_{ab}u^a(\sigma)u^b(\sigma)d\sigma \tag{10}$$

are the equations of hydrodynamical type: $\dot u^a(\tau,\sigma)=u'^a(\tau,\sigma)$. Any system from the hierarchy

$$\frac{\partial u^a(\tau,\sigma)}{\partial\tau_n}=\int_0^\pi(R(\sigma,\sigma_1)R(\sigma_1,\sigma_2)...R(\sigma_{n-1},\sigma_n))^a_b(u)\frac{\partial u^b(\tau_n,\sigma_n)}{\partial\sigma_n}d\sigma_1d\sigma_2...d\sigma_n,Z\ni n. \tag{11}$$

is a multi-Hamiltonian integrable system. In particular, any closed string model of the form

$$\frac{\partial u^a(\sigma)}{\partial \tau_1} = \int_0^\pi R_b^a(\sigma,\sigma')\frac{\partial u(\sigma')}{\partial \sigma'}d\sigma' \tag{12}$$

is integrable system:

$$\frac{\partial u^a(\sigma)}{\partial \tau_1} = \frac{d}{d\sigma}[F^a(u) + \eta^{ak}\eta_{bl}\frac{\partial F^b}{\partial u^k}u^l - \eta^{ak}\sum_{\alpha=1}^{L}\frac{\partial \psi^\alpha}{\partial u^k}\psi^\alpha], \tag{13}$$

where functions $F^a(u), \psi^\alpha(u)$ are arbitrary solutions of the equations (8). The equation (12) can be obtaiin as Hamiltonian equation of motion with the Hamiltonian

$$H_2 = \int_0^\pi [\eta_{ab}F^b(u(\sigma))u^a(\sigma) - \frac{1}{2}\sum_{\alpha=1}^{L}\psi^\alpha(u)\psi^\alpha(u)]d\sigma. \tag{14}$$

The brackets (5), (7) and Hamiltonians (10), (14) satisfy bi-Hamiltonity conditions (1).

Let us consider the constant curvature manifold. Mokhov, Ferapontov [2] constructed nonlocal PB which it exactly corresponds to an D-mensional surface with constant Riemannian curvature K, embedded in a pseudo-Euclidean space E^{D+1}. It has following form:

$$\{u^a(\sigma),u^b(\sigma')\}_{MF} = g^{ab}(u(\sigma))\frac{d}{d\sigma}\delta(\sigma-\sigma') - g^{ac}(u)\Gamma_{ck}^b(u)u'^k(\sigma)\delta(\sigma-\sigma') + K\frac{\partial u^a(\sigma)}{\partial\sigma}\nu(\sigma'-\sigma)\frac{\partial u^b(\sigma')}{\partial\sigma'}. \tag{15}$$

It was constructed in [9-11] compatible pairs of constant PB (5) and Mokhov, Ferapontov PB (15):

$$\{u^a(\sigma),u^b(\sigma')\}_{MF} = P_{MF}^{ab}(\sigma,\sigma')(u) = [\eta^{ac}\frac{\partial h^b(u(\sigma))}{\partial u^c} + \eta^{bc}\frac{\partial h^a(u(\sigma))}{\partial u^c} - Ku^a u^b]\frac{d}{d\sigma}\delta(\sigma-\sigma')$$
$$+[\eta^{ac}\frac{\partial^2 h^b(u(\sigma))}{\partial u^c\partial u^k} - K\delta_k^a u^b(\sigma)]\frac{\partial u^k(\sigma)}{\partial\sigma}\delta(\sigma-\sigma') + K\frac{\partial u^a(\sigma)}{\partial\sigma}\nu(\sigma'-\sigma)\frac{\partial u^b(\sigma')}{\partial\sigma'},$$

where functions $h^a(u(\sigma))$ satisfy the equations:

$$\frac{\partial^2 h^a}{\partial u^k\partial u^l}\eta^{\ln}\frac{\partial^2 h^b}{\partial u^n\partial u^c} = \frac{\partial^2 h^b}{\partial u^k\partial u^l}\eta^{\ln}\frac{\partial^2 h^a}{\partial u^n\partial u^c}, g^{ak}{}_{can}\eta^{bl}\frac{\partial^2 h^a}{\partial u^k\partial u^l} = g^{bk}{}_{can}\eta^{al}\frac{\partial^2 h^a}{\partial u^k\partial u^l},$$
$$g^{ab}{}_{can} = \eta^{ac}\frac{\partial h^b}{\partial u^c} + \eta^{bc}\frac{\partial h^a}{\partial u^c} - Ku^a u^b. \tag{16}$$

The recursion operator, constructed from Hamiltonian operators $P_{MF}^{ab}(\sigma,\sigma')(u), P_1^{ab}(\sigma,\sigma')(u)$ leds to a multi-Hamiltonian integrable system (11). In particular, any system of the form (12) is system of hydrodynamical type with Hamiltonian H_2 of the following form:

$$H_2 = \int_0^\pi [\eta_{ab}h^b(u(\sigma))u^a(\sigma) - \frac{K}{8}\eta_{ab}u^a u^b\eta_{kl}u^k u^l]d\sigma, \tag{17}$$

where functions $h^a(u(\sigma))$ are arbitrary solutions of the equations (16).

CONCLUSIONS

The closed string model is considered as a bi-Hamiltonian system of hydrodynamical type. A general nonlocal PB of Ferapontov [4] and Mokhov, Ferapontov nonlocal PB [5] for constant curvature surface were used to construct nonlinear integrable string models under string coordinates $u^a(\tau,\sigma) = \frac{1}{\sqrt{2}}[\eta^{ab}p_b(\tau,\sigma) + \frac{\partial x^a(\tau,\sigma)}{\partial\sigma}]$. The recursion operator was obtained. Second string Hamiltonian H_2 was obtained.

ACKNOWLEDGMENT

Author deeps thank Prof. Stephan Duplij for invitation to participate in the International conference “Karazin natural studioos”.

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ИНТЕГРИРУЕМЫЕ СТРУННЫЕ МОДЕЛИ ГИДРОДИНАМИЧЕСКОГО ТИПА

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Модель замкнутой струны рассмотрена как бигамильтонова система гидродинамического типа. Нелокальные скобки Пуассона были использованы для построения интегрируемых нелинейных струнных моделей не сигма-модельного типа. Рекурсионный оператор был получен. Второй струнный гамильтониан был получен.
КЛЮЧЕВЫЕ СЛОВА: струнная модель, интегрируемая модель, нелокальные скобки, гидродинамического типа модель