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COINCIDENCE LIMIT AND GENERALIZED INTERACTION TERM STRUCTURE IN MULTIGRAVITY

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The generalized structure of the interaction term of multigravity is analyzed in detail. The coincidence limit of any multigravity theory is defined and the compatibility equation for the interaction potential is derived which is studied in the weak perturbation limit of metric. The most general properties of the invariant volume and the scalar potential of multigravity are investigated. The general formula for multigravity invariant volume using three means (arithmetic, geometric and harmonic) is derived. The Pauli-Fierz mass term for bigravity in the weak field limit is obtained.
KEY WORDS: coincidence limit, compatibility equation, invariant volume, weak field limit, scalar potential

The multigravity extension of General Relativity (in first papers it was called "*f*-*g* theory" or "strong gravity" [1–3]) is important both from theoretical constructions (quantum gravity and branes [4–6], discrete dimensions [7,8], renormalization [9], massive gravity [10] etc.) and experimental facts (dark matter and dark energy [11–13], cosmic acceleration [14, 15] etc.). In this respect it is worthwhile to consider non-linear formulation of multigravity [16]. The shape of interaction term plays the most crucial role in constructing models.

The goal of this paper is to consider the generalized structure of the interaction term in detail (see also [17]). That is, we introduce the coincidence limit of a multigravity theory and obtain the compatibility equation for the interaction potential and analyze it in the weak perturbation limit. Note that a particular case of our general construction, a "perturbative limit" which corresponds to critical points of interacting potential and depends from their special form of interaction potential, was considered in [16] for bigravity only. Here we propose the multigravity generalizations and do not consider any restrictions on the metric, as in [16] (where only spaces with constant curvature were considered).

Also we study the most general properties of invariant volume in the interaction term and the scalar potential of multigravity. We generalize the invariant volume for multigravity for three means and obtain the Pauli-Fierz mass term [18] for bigravity in the weak field limit [16], as an example.

MULTIGRAVITY AND THE COINCIDENCE LIMIT

We consider several Universes (labelled by $i = 1, \dots N$) each described by the metric $\mathrm{g}^{(i)}_{\mu\nu}$ (we use the signature $+---$), the set of matter fields $\Phi^{(i)}$ (scalar, spinorial, vector ones) and the action

$$S_{G(i)} = \int d\Omega^{(i)} \left[F^{(i)}(\mathrm{g}^{(i)}) + L\left(\mathrm{g}^{(i)}, \Phi^{(i)}\right)\right], \tag{1}$$

where $d\Omega^{(i)} = d^4x\sqrt{g^{(i)}}$, $g^{(i)} = -\det\left(\mathrm{g}^{(i)}_{\mu\nu}\right) > 0$ (distinguishing $g^{(i)}$ as a positive number and $\mathrm{g}^{(i)}$ as a tensor) is the invariant volume and $F^{(i)}(\mathrm{g}^{(i)})$ is pure gravity Lagrangian of i-Universe, $L\left(\mathrm{g}^{(i)}, \Phi^{(i)}\right)$ describes coupling of matter fields and gravity. In the concept of Weakly Coupled Worlds [16] due to the no-go theorem of [19] the only consistent nonlinear theory of N massless gravitons is the sum of decoupled gravity actions (1)

$$S_0 = \sum_i^N S_{G(i)} \tag{2}$$

which has the huge symmetry $\prod_i^N \mathrm{diff}_{(i)}$ (each $\mathrm{diff}_{(i)}$ acts on its metric $\mathrm{g}^{(i)}_{\mu\nu}$ and matter fields $\Phi^{(i)}$). The full action of multigravity, as Weakly Coupled Worlds mixing by their gravitational fields only, is

$$S_{mG} = \sum_i^N \int d\Omega^{(i)} \left[F^{(i)}(\mathrm{g}^{(i)}) + L\left(\mathrm{g}^{(i)}, \Phi^{(i)}\right)\right] + \int d^4x W(\mathrm{g}^{(1)}, \mathrm{g}^{(2)}, \dots \mathrm{g}^{(N)}), \tag{3}$$

where $W(\mathrm{g}^{(1)}, \mathrm{g}^{(2)}, \dots \mathrm{g}^{(N)})$ is the interaction term which is a scalar density made up from metrics taken at the same point, i.e. in ultralocal limit [16]. The symmetry of (3) reduces to only one diffeomorphism, because of the no-go theorem [19].

Therefore, it is interesting to consider the case when also the Universities are described by the same metric. So let us introduce the coincidence limit, when $\mathsf{g}^{(1)}_{\mu\nu} = \mathsf{g}^{(2)}_{\mu\nu} = \ldots = \mathsf{g}^{(N)}_{\mu\nu} \equiv \mathsf{g}_{\mu\nu}$. In case of the absence of interaction ($W = 0$) and matter, we have

$$S_0 = \int d\Omega \sum_i^N F^{(i)}(\mathsf{g}). \tag{4}$$

If $F^{(1)}(\mathsf{g}) = F^{(2)}(\mathsf{g}) = \ldots = F^{(N)}(\mathsf{g}) \equiv F(\mathsf{g})$, then $S_0 = N \int d\Omega F(\mathsf{g}_{\mu\nu})$, and therefore noninteracting full theory coincides with the initial one. But in the case of interacting theory and moreover nonvanishing interacting term in the coincidence limit the multigravity can be equivalent to some effective gravity theory described by the effective metric $\tilde{\mathsf{g}}_{\mu\nu}$ and effective function $\tilde{F}\left(\tilde{\mathsf{g}}\right)$. Thus we arrive to the compatibility equation

$$\sqrt{g}\left(F(\mathsf{g}) + U\left(\mathsf{g}\right)\right) = \sqrt{\tilde{g}} F(\tilde{\mathsf{g}}), \tag{5}$$

where $\sqrt{g} U\left(\mathsf{g}\right) = W(\mathsf{g}, \mathsf{g}, \ldots \mathsf{g}) \neq 0$, and all functions are taken in the same 'point'. The equation (5) is defined up to covariant divergence of any function, because it will not contribute to the equations of motion. In [1, 2, 16] the only case $W(\mathsf{g}, \mathsf{g}, \ldots \mathsf{g}) = 0$ ($U\left(\mathsf{g}\right) = 0$) was considered, and the compatibility equation has the trivial solution $\tilde{\mathsf{g}} = \mathsf{g}$ only. Here we extend the consideration to nonvanishing $U\left(\mathsf{g}\right)$, which allows us to obtain possible nontrivial solutions. The physical sense of the compatibility equation (5) is treatment of two equal interacting Universes (having the same function F) in the limit of coinciding metric tensors, as some "effective" Universe described by this function F, but another metric tensor $\tilde{\mathsf{g}}$.

In general case the formal solution of the compatibility equation (5) can be presented as

$$\tilde{\mathsf{g}}_{\mu\nu} = \Phi_{\mu\nu}\left(\mathsf{g}, U\left(\mathsf{g}\right)\right),$$

where the function $\Phi_{\mu\nu}$ is a symmetric covariant tensor determining the transformation $\mathsf{g}_{\mu\nu} \to \tilde{\mathsf{g}}_{\mu\nu}$.

Let us solve the compatibility equation in the simplest case: small fields expansion

$$\tilde{\mathsf{g}}_{\mu\nu} = \mathsf{g}_{\mu\nu} + \mathsf{p}_{\mu\nu}. \tag{6}$$

We note that here we consider $\mathsf{g}_{\mu\nu}$ as an arbitrary metric, but not necessarily flat space metric $\mathsf{g}_{\mu\nu} \neq \eta_{\mu\nu}$. In the first order of $\mathsf{p}_{\mu\nu}$ for determinants we derive

$$\begin{aligned} \det\left(\tilde{\mathsf{g}}\right) &= \det\left(\mathsf{g}\right) + \mathsf{p}_{\alpha\beta}\mathsf{K}^{\alpha\beta}\left(\mathsf{g}\right), \\ \mathsf{K}^{\alpha\beta}\left(\mathsf{g}\right) &= \varepsilon^{\mu\nu\rho\sigma}\left(\delta^\alpha_0\delta^\beta_\mu \mathsf{g}_{1\nu}\mathsf{g}_{2\rho}\mathsf{g}_{3\sigma} + \delta^\alpha_1\delta^\beta_\nu \mathsf{g}_{0\mu}\mathsf{g}_{2\rho}\mathsf{g}_{3\sigma} + \delta^\alpha_2\delta^\beta_\rho \mathsf{g}_{0\mu}\mathsf{g}_{1\nu}\mathsf{g}_{3\sigma} + \delta^\alpha_3\delta^\beta_\sigma \mathsf{g}_{0\mu}\mathsf{g}_{1\nu}\mathsf{g}_{2\rho}\right). \end{aligned} \tag{7}$$

If we consider expansion around Minkowski metric $\mathsf{g}_{\mu\nu} = \eta_{\mu\nu}$, then $\mathsf{K}^{\alpha\beta}\left(\mathsf{g}\right) = -\eta^{\alpha\beta}$ and $\tilde{g} = -\det\left(\tilde{\mathsf{g}}\right) = 1 + \mathrm{Tr}\,\mathsf{p}$, where $\mathrm{Tr}\,\mathsf{p} \equiv \mathsf{p}_{\alpha\beta}\eta^{\alpha\beta}$. In general case, after substitution of (7) into the main compatibility equation (5), we obtain

$$U\left(\mathsf{g}\right) = \left(\frac{\partial F\left(\mathsf{g}\right)}{\partial \mathsf{g}_{\mu\nu}} - \frac{1}{2\sqrt{g}} F\left(\mathsf{g}\right)\mathsf{K}^{\mu\nu}\left(\mathsf{g}\right)\right)\mathsf{p}_{\mu\nu} + \frac{\partial F\left(\mathsf{g}\right)}{\partial \mathsf{g}_{\mu\nu,\rho}}\mathsf{p}_{\mu\nu,\rho} + \frac{\partial F\left(\mathsf{g}\right)}{\partial \mathsf{g}_{\mu\nu,\rho\sigma}}\mathsf{p}_{\mu\nu,\rho\sigma} + \ldots, \tag{8}$$

where "…" denote similar derivatives by higher than two derivatives of $\mathsf{g}_{\mu\nu}$ terms.

So any multigravity model (1) induces the interaction term which in the coincidence limit has the form (8). On the other hand, the relation (8) can be considered as an equation for $\mathsf{p}_{\mu\nu}$, and therefore we can determine an effective metric $\tilde{\mathsf{g}}_{\mu\nu}$ of gravity theory, which is equivalent to a given multigravity in the coincidence limit, for any interaction term.

In most cases $F\left(\mathsf{g}\right)$ is a function of Riemann curvature $R_{\mu\nu\rho\sigma}\left(\mathsf{g}\right)$ which contains only up to 2 derivatives of the metric, and so the higher terms in (8) denoted by … will not appear. The most general polynomial shape of such $F\left(\mathsf{g}\right)$ is

$$F\left(\mathsf{g}\right) = \hat{F}\left(R_{\mu\nu\rho\sigma}\left(\mathsf{g}\right)\right) = A \cdot R^n\left(\mathsf{g}\right) + B \cdot R^m_{\mu\nu}\left(\mathsf{g}\right) + C \cdot R^r_{\mu\nu\rho\sigma}\left(\mathsf{g}\right), \tag{9}$$

where A, B, C are constants and

$$R^\mu{}_{\nu\rho\sigma}\left(\mathsf{g}\right) = \Gamma^\mu{}_{\nu\sigma,\rho}\left(\mathsf{g}\right) - \Gamma^\mu{}_{\nu\rho,\sigma}\left(\mathsf{g}\right) + \Gamma^\mu{}_{\tau\rho}\left(\mathsf{g}\right)\Gamma^\tau{}_{\nu\sigma}\left(\mathsf{g}\right) - \Gamma^\mu{}_{\tau\sigma}\left(\mathsf{g}\right)\Gamma^\tau{}_{\nu\rho}\left(\mathsf{g}\right), \tag{10}$$

$$\Gamma^\mu{}_{\nu\rho}\left(\mathsf{g}\right) = \frac{1}{2}\mathsf{g}^{\mu\sigma}\left(\mathsf{g}_{\sigma\nu,\rho} + \mathsf{g}_{\sigma\rho,\nu} - \mathsf{g}_{\nu\rho,\sigma}\right), \tag{11}$$

$$R_{\mu\nu}\left(\mathsf{g}\right) = R^\rho{}_{\mu\rho\nu}\left(\mathsf{g}\right), \qquad R\left(\mathsf{g}\right) = \mathsf{g}^{\mu\nu}R_{\mu\nu}\left(\mathsf{g}\right). \tag{12}$$

The standard Einstein gravity corresponds to $F\left(\mathsf{g}\right) = A_{Einsten} \cdot R\left(\mathsf{g}\right)$ [20]. In this case and using (6) we have (note the absence of the first derivatives of $\mathsf{g}_{\mu\nu}$)

$$U_{Einsten}\left(\mathsf{g}\right) = A_{Einsten}\left[\left(\frac{\partial R\left(\mathsf{g}\right)}{\partial \mathsf{g}_{\mu\nu}} - \frac{1}{2\sqrt{g}} R\left(\mathsf{g}\right)\mathsf{K}^{\mu\nu}\left(\mathsf{g}\right)\right)\mathsf{p}_{\mu\nu} + \frac{\partial R\left(\mathsf{g}\right)}{\partial \mathsf{g}_{\mu\nu,\rho}}\mathsf{p}_{\mu\nu,\rho} + \frac{\partial R\left(\mathsf{g}\right)}{\partial \mathsf{g}_{\mu\nu,\rho\sigma}}\mathsf{p}_{\mu\nu,\rho\sigma}\right]. \tag{13}$$

It is convenient to use covariant derivatives by $\mathsf{g}_{\mu\nu}$, then

$$\widetilde{\Gamma}^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\nu\rho} + \frac{1}{2}\mathsf{g}^{\mu\sigma}(\mathsf{p}_{\sigma\nu;\rho} + \mathsf{p}_{\sigma\rho;\nu} - \mathsf{p}_{\nu\rho;\sigma}), \tag{14}$$

$$\widetilde{R}^{\mu}_{\nu\rho\sigma} = R^{\mu}_{\nu\rho\sigma} + \frac{1}{2}\mathsf{g}^{\mu\alpha}(\mathsf{p}_{\alpha\nu;\sigma\rho} + \mathsf{p}_{\alpha\sigma;\nu\rho} - \mathsf{p}_{\nu\sigma;\alpha\rho} - \mathsf{p}_{\alpha\nu;\rho\sigma} - \mathsf{p}_{\alpha\rho;\nu\sigma} + \mathsf{p}_{\nu\rho;\alpha\sigma}), \tag{15}$$

$$\widetilde{R} \equiv \widetilde{g}^{\nu\sigma}\widetilde{R}^{\mu}_{\nu\mu\sigma} = R - \mathsf{p}^{\alpha\beta}R_{\alpha\beta} - \Box\mathsf{p}^{\alpha}_{\alpha;\beta}{}^{\beta} + \frac{1}{2}\mathsf{g}^{\alpha\beta}\mathsf{g}^{\mu\nu}(\mathsf{p}_{\beta\mu;\nu\alpha} - \mathsf{p}_{\beta\mu;\alpha\nu} + \mathsf{p}_{\beta\nu;\mu\alpha} + \mathsf{p}_{\mu\alpha;\beta\nu}), \tag{16}$$

where $\Box$ is covariant D'Alambertian defined as $\Box = \nabla_\mu\nabla^\mu$ and ∇_μ is covariant derivative by $\mathsf{g}_{\mu\nu}$, i.e. $\Box\mathsf{p} \equiv \mathsf{p}^{\alpha}_{\alpha;\beta}{}^{\beta}$.

After substitution to (5) we obtain

$$\int \widetilde{R}\sqrt{\widetilde{g}}d^4x = \int \sqrt{g}d^4xR$$
$$+\int \sqrt{g}d^4x\left[-\mathsf{p}^{\alpha\beta}R_{\alpha\beta} - \Box\mathsf{p} + \frac{1}{2}\mathsf{g}^{\alpha\beta}\mathsf{g}^{\mu\nu}(\mathsf{p}_{\beta\mu;\nu\alpha} - \mathsf{p}_{\beta\mu;\alpha\nu} + \mathsf{p}_{\beta\nu;\mu\alpha} + \mathsf{p}_{\mu\alpha;\beta\nu}) - \frac{R}{2\sqrt{g}}\mathsf{p}_{\alpha\beta}\mathsf{F}^{\alpha\beta}\right]. \tag{17}$$

GENERALIZED INVARIANT VOLUME IN MULTIGRAVITY

In consideration of the interaction term of multigravity it is important to choose consistently the invariant volume which in coincidence limit transforms to the standard invariant volume $d^4x\sqrt{g}$. For simplicity, first we consider the bigravity case [16].

Note that $d^4xW(\mathsf{g}^{(1)},\mathsf{g}^{(2)})$ is a scalar, while d^4x and $W(\mathsf{g}^{(1)},\mathsf{g}^{(2)})$ are the scalar densities of opposite weights. By analogy with usual invariant volume $d\Omega = d^4x\sqrt{g}$, we can present $d^4xW(\mathsf{g}^{(1)},\mathsf{g}^{(2)})$ as a product $d^4x \cdot f\left(\sqrt{g_1},\sqrt{g_2}\right) \cdot V(\mathsf{g}^{(1)},\mathsf{g}^{(2)})$, where $V(\mathsf{g}^{(1)},\mathsf{g}^{(2)})$ is a scalar interaction potential.

Now we demand that the 'interaction' invariant volume $d\Omega_{int} = d^4xf\left(\sqrt{g_1},\sqrt{g_2}\right)$ should be a scalar which in the coincidence limit $\mathsf{g}^{(1)}_{\mu\nu} = \mathsf{g}^{(2)}_{\mu\nu} \equiv \mathsf{g}_{\mu\nu}$ gives the standard invariant volume $d\Omega_{int} \to d\Omega$. To satisfy these conditions we require the following general properties of the function $f(u,v)$:

1) Idempotence $f(u,u) = u$; 2) Monotony; 3) Homogeneity $f(tu,tv) = tf(u,v)$; 4) Symmetry $f(u,v) = f(v,u)$.

From homogeneity and symmetry it follows that $f(u,v)$ can be expressed through the function of one variable, the ratio $\frac{u}{v}$, as

$$f(u,v) = u \cdot f\left(\frac{v}{u},1\right) = v \cdot f\left(\frac{u}{v},1\right) = \sqrt{uv} \cdot f\left(\sqrt{\frac{u}{v}},\sqrt{\frac{v}{u}}\right). \tag{18}$$

Thus, the interaction invariant volume can be presented as

$$d\Omega_{int} = d^4xf\left(\sqrt{g_1},\sqrt{g_2}\right) = d^4x \cdot \sqrt[4]{g_1g_2} \cdot f\left(\sqrt[4]{\frac{g_1}{g_2}},\sqrt[4]{\frac{g_2}{g_1}}\right) = d^4x \cdot \sqrt[4]{g_1g_2} \cdot \hat{f}\left(\frac{g_2}{g_1}\right). \tag{19}$$

From symmetry of $f(u,v)$ it follows that $\hat{f}(u) = \hat{f}\left(u^{-1}\right)$.

Let us consider an example. The simplest functions satisfying (18) are usual averages: arithmetic mean, harmonic mean and geometric mean[1]. It is reasonable to consider their linear combination, which gives for the generalized 'interaction' invariant volume the following expression

$$\begin{aligned} d\Omega_{int}(a,b,c) &= d^4xf\left(\sqrt{g_1},\sqrt{g_2}\right) = \frac{d^4x}{a+b+c}\left(a\frac{\sqrt{g_1}+\sqrt{g_2}}{2} + b\sqrt[4]{g_1g_2} + c\frac{2}{\frac{1}{\sqrt{g_1}}+\frac{1}{\sqrt{g_2}}}\right) \\ &= d^4x \cdot \sqrt[4]{g_1g_2} \cdot \frac{1}{a+b+c}\left[\frac{a}{2}\left(\sqrt[4]{\frac{g_1}{g_2}}+\sqrt[4]{\frac{g_2}{g_1}}\right) + b + 2c\frac{1}{\sqrt[4]{\frac{g_1}{g_2}}+\sqrt[4]{\frac{g_2}{g_1}}}\right] \\ &= d^4x \cdot \sqrt[4]{g_1g_2} \cdot \frac{1}{a+b+c}\left[\frac{a}{2}\left(y+\frac{1}{y}\right) + b + 2c\frac{1}{y+\frac{1}{y}}\right], \end{aligned} \tag{20}$$

[1] Usually one considers the geometric mean only (e.g. see [16]).

where a, b, c are arbitrary real constants and $y=\sqrt[4]{\frac{g_1}{g_2}}$. Similar formulas are valid for N-multigravity

$$d\Omega_{int}=d^4x\cdot f\left(\sqrt{g_1},...,\sqrt{g_N}\right)=d^4x\ \sqrt[2N]{g_1...g_N}\cdot f\left(\sqrt[2N]{\frac{g_1^{N-1}}{g_2g_3...g_N}},\ \sqrt[2N]{\frac{g_2^{N-1}}{g_1g_3...g_N}}...,\ \sqrt[2N]{\frac{g_N^{N-1}}{g_1g_2...g_{N-1}}}\right). \tag{21}$$

Evidently, this formula for $N=2$ (bigravity) gives (19).

Let us denote the N arguments of the function f as

$$y_1^{(N)}=\sqrt[2N]{g_1^{N-1}g_2^{-1}g_3^{-1}...g_N^{-1}},y_2^{(N)}=\sqrt[2N]{g_1^{-1}g_2^{N-1}g_3^{-1}...g_N^{-1}},\ldots\quad y_N^{(N)}=\sqrt[2N]{g_1^{-1}g_2^{-1}...g_{N-1}^{-1}g_N^{N-1}}, \tag{22}$$

which obviously satisfy

$$y_1^{(N)}\cdot y_2^{(N)}\cdot\ldots\cdot y_N^{(N)}=1. \tag{23}$$

Therefore the function f has actually $N-1$ independent arguments, and so

$$d\Omega_{int}=d^4x\cdot f\left(\sqrt{g_1},...,\sqrt{g_N}\right)=d^4x\ \sqrt[2N]{g_1...g_N}\cdot\hat{f}\left(y_1^{(N)},y_2^{(N)},\ldots y_N^{(N-1)}\right), \tag{24}$$

which for $N=2$ gives (19). Using the means as in (20) we obtain its N-analog

$$d\Omega_{int}=d^4x\cdot\ \sqrt[2N]{g_1...g_N}\cdot\frac{1}{a+b+c}\left[\frac{a}{N}\sum_{i=1}^{N}y_i^{(N)}+b+c\frac{N}{\sum_{i=1}^{N}\frac{1}{y_i^{(N)}}}\right], \tag{25}$$

which can be considered as the most general 'interaction' invariant volume for multigravity.

GENERALIZED INTERACTION POTENTIAL IN MULTIGRAVITY

Let us construct the most general expression for the interaction term in (3)

$$S_{int}=\int d^4xW(\mathsf{g}^{(1)},\mathsf{g}^{(2)},\ldots\mathsf{g}^{(N)}). \tag{26}$$

It is convenient to extract the generalized invariant volume (presented in previous section*)

$$S_{int}=\int d\Omega_{int}V(\mathsf{g}^{(1)},\mathsf{g}^{(2)},\ldots\mathsf{g}^{(N)}), \tag{27}$$

where $V(\mathsf{g}^{(1)},\mathsf{g}^{(2)},\ldots\mathsf{g}^{(N)})$ is the scalar interaction potential of multigravity.

As we noted before, the symmetry of the full action (3) can be reduced to only one diffeomorphism group which is the diagonal subgroup of common diffeomorphisms acting on metrics as Lie derivative $\delta\mathsf{g}^{(i)}=\mathcal{L}_\varepsilon\mathsf{g}^{(i)}$ or in manifest form

$$\delta\mathsf{g}_{\mu\nu}^{(i)}=\varepsilon^\rho\mathsf{g}_{\mu\nu,\rho}^{(i)}+\varepsilon^\rho{}_{,\mu}\mathsf{g}_{\rho\nu}^{(i)}+\varepsilon^\rho{}_{,\nu}\mathsf{g}_{\mu\rho}^{(i)}, \tag{28}$$

where ε^ρ is the same for all metrics. This symmetry restricts the shape of the scalar interaction potential: it should depend from invariant which can be constructed from N metrics $\mathsf{g}_{\mu\nu}^{(i)}$.

Let us consider bigravity as an example [14, 16]. The scalar potential should depend from invariants of the mixed tensor

$$\mathsf{Y}_\nu^\mu=\mathsf{g}_{\nu\rho}^{(1)}\mathsf{g}^{(2)\rho\mu}, \tag{29}$$

which can be treated as tensorial analog of the scalar variable y from (20). Note that Y_ν^μ is diffeomorphism invariant, i.e. under transformation (28) we have $\delta\mathsf{Y}=\mathcal{L}_\varepsilon\mathsf{Y}$, because of the same ε^ρ for all metrics. To calculate invariants of the tensor (29) we take powers of traces of the the matrix Y corresponding to the tensor Y_ν^μ, and the number of invariants in 4 dimensions is 4 by the Cayley theorem, which can be taken as

$$\varkappa_1=\mathrm{Tr}\,Y,\quad\varkappa_2=\mathrm{Tr}\,Y^2,\quad\varkappa_3=\mathrm{Tr}\,Y^3,\quad\varkappa_4=\mathrm{Tr}\,Y^4. \tag{30}$$

Let $\lambda_{(i)}$ $(i=0,1,2,3)$ are eigenvalues of the tensor Y_ν^μ, which can be treated as relative eigenvalues of the metric $\mathsf{g}^{(1)}$ relatively $\mathsf{g}^{(2)}$. In the special bi-orthogonal vierbein $e_\mu^{(i)}$ the metrics can be written as follows

$$\mathsf{g}_{\mu\nu}^{(1)}=\lambda_{(0)}e_\mu^{(0)}e_\nu^{(0)}-\lambda_{(1)}e_\mu^{(1)}e_\nu^{(1)}-\lambda_{(2)}e_\mu^{(2)}e_\nu^{(2)}-\lambda_{(3)}e_\mu^{(3)}e_\nu^{(3)}, \tag{31}$$

$$\mathsf{g}_{\mu\nu}^{(2)}=e_\mu^{(0)}e_\nu^{(0)}-e_\mu^{(1)}e_\nu^{(1)}-e_\mu^{(2)}e_\nu^{(2)}-e_\mu^{(3)}e_\nu^{(3)}, \tag{32}$$

and so the matrix Y is diagonal. In case of real and positive eigenvalues $\lambda_{(i)}$ it is convenient to introduce

$$\mu_{(i)} = \ln \lambda_{(i)} \tag{33}$$

and consider their powers

$$\sigma_1 = \mu_{(0)} + \mu_{(1)} + \mu_{(2)} + \mu_{(3)}, \quad \sigma_2 = \mu_{(0)}^2 + \mu_{(1)}^2 + \mu_{(2)}^2 + \mu_{(3)}^2, \tag{34}$$

$$\sigma_3 = \mu_{(0)}^3 + \mu_{(1)}^3 + \mu_{(2)}^3 + \mu_{(3)}^3, \quad \sigma_4 = \mu_{(0)}^4 + \mu_{(1)}^4 + \mu_{(2)}^4 + \mu_{(3)}^4. \tag{35}$$

Then the scalar potential of bigravity is a function of the introduced invariants σ_n as

$$V(\mathsf{g}^{(1)}, \mathsf{g}^{(2)}) = \hat{V}(\sigma_1, \sigma_2, \sigma_3, \sigma_4). \tag{36}$$

An important class of bigravity models has the symmetry $\mathsf{g}^{(1)} \leftrightarrow \mathsf{g}^{(2)}$. In this case for eigenvalues we have $\lambda_{(i)} \to \lambda_{(i)}^{-1}$ and $\mu_{(i)} \to -\mu_{(i)}$, therefore

$$\hat{V}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \hat{V}(-\sigma_1, \sigma_2, -\sigma_3, \sigma_4). \tag{37}$$

In the weak field limit $\sigma_i \to 0$ it is sufficient to take into account only two first invariants σ_1, σ_2 and consider $\hat{V}_0(\sigma_1, \sigma_2) = \hat{V}(\sigma_1, \sigma_2, 0, 0)$, which appears naturally in brane models [6] and "Pauli-Fierz-like" bigravity [14]. For the latter we expand

$$\mathsf{g}_{\mu\nu}^{(1)} = \eta_{\mu\nu} + \sqrt{2k_1}\mathsf{h}_{\mu\nu}^{(1)}, \quad \mathsf{g}_{\mu\nu}^{(2)} = \eta_{\mu\nu} + \sqrt{2k_2}\mathsf{h}_{\mu\nu}^{(2)}, \tag{38}$$

where $\eta_{\mu\nu}$ is the same flat metric. In this limit the mixed tensor (29) is

$$\mathsf{Y}_\nu^\mu = \delta_\nu^\mu + \sqrt{2k_1}\mathsf{h}_\nu^{(1)\mu} - \sqrt{2k_2}\mathsf{h}_\nu^{(2)\mu}. \tag{39}$$

Let us consider the combinations

$$\mathsf{h}_{\mu\nu}^0 = q_1\mathsf{h}_{\mu\nu}^{(2)} + q_2\mathsf{h}_{\mu\nu}^{(1)}, \qquad \mathsf{h}_{\mu\nu}^{mass} = q_1\mathsf{h}_{\mu\nu}^{(2)} - q_2\mathsf{h}_{\mu\nu}^{(1)}, \tag{40}$$

where $q_1^2 + q_2^2 = 1$, then it can be shown that $\mathsf{h}_{\mu\nu}^0$ is massless and $\mathsf{h}_{\mu\nu}^{mass}$ contains the Pauli-Fierz term. Indeed,

$$\sigma_1 = \sqrt{2k_1}\mathsf{h}_\mu^{(1)\mu} - \sqrt{2k_2}\mathsf{h}_\mu^{(2)\mu} + k_2\mathsf{h}_{\mu\nu}^{(2)}\mathsf{h}^{(2)\mu\nu} - k_1\mathsf{h}_{\mu\nu}^{(1)}\mathsf{h}^{(1)\mu\nu}, \tag{41}$$

$$\sigma_2 = 2k_1\mathsf{h}_{\mu\nu}^{(1)}\mathsf{h}^{(1)\mu\nu} + 2k_2\mathsf{h}_{\mu\nu}^{(2)}\mathsf{h}^{(2)\mu\nu} - 4\sqrt{k_1k_2}\mathsf{h}_{\mu\nu}^{(1)}\mathsf{h}^{(2)\mu\nu}. \tag{42}$$

Finally we obtain

$$\mathsf{h}_{\mu\nu}^{mass}\mathsf{h}^{mass,\mu\nu} - \left(\mathsf{h}_\mu^{mass,\mu}\right)^2 = \frac{1}{2(k_1+k_2)}\left(\sigma_2 - \sigma_1^2\right). \tag{43}$$

Thus, if we choose the interaction in the form

$$S_{int} = -\frac{1}{k_1+k_2}\int d\Omega_{int}\hat{V}_0(\sigma_1, \sigma_2), \tag{44}$$

where $d\Omega_{int}$ is defined in (20) and the scalar interaction potential is

$$\hat{V}_0^{PF}(\sigma_1, \sigma_2) = \frac{m_{PF}^2}{8}\left(\sigma_2 - \sigma_1^2\right), \tag{45}$$

then the weak field limit of bigravity generates the Pauli-Fierz mass term [18] of the shape

$$S_{int} = -\frac{m_{PF}^2}{4}\int d^4x \left(\mathsf{h}_{\mu\nu}^{mass}\mathsf{h}^{mass,\mu\nu} - \left(\mathsf{h}_\mu^{mass,\mu}\right)^2\right). \tag{46}$$

For the brane motivated bigravity scenario [4, 21] the scalar potential has the form [16]

$$\hat{V}_0^{brane}(\sigma_1, \sigma_2) = m^2\left(\cosh\frac{\sigma_1}{4} - \cosh\frac{\sqrt{4\sigma_2 - \sigma_1^2}}{4\sqrt{3}}\right). \tag{47}$$

In the weak field limit it reproduces the Pauli-Fierz mass term (45), indeed

$$\hat{V}_0^{brane}(\sigma_1, \sigma_2)\,|_{m=\sqrt{3}m_{PF}} = \hat{V}_0^{PF}(\sigma_1, \sigma_2). \tag{48}$$

Note that the "perturbative limit" which corresponds to existence of critical point of potential and from which for bigravity (with potential of form (29) only) it follows that $\mathsf{g}_{\mu\nu}^{(1)} = \mathsf{g}_{\mu\nu}^{(2)}$, was considered in [16]. Here we present a more general case which is not connected with any concrete form of the interaction potential and does not demand consideration of spaces with constant curvature.

CONCLUSIONS

So in this paper we have analyzed the generalized structure of the interaction term of multigravity. We introduced the coincidence limit and obtained the compatibility equation for the interaction potential which was studied in the weak perturbation limit. We considered the most general properties of invariant volume and the scalar potential. As an example, we derived the Pauli-Fierz mass term for bigravity in the weak field limit.

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REFERENCES

1. Isham C. J., Salam A., Strathdee J. f-Dominance of gravity // Phys. Rev. - 1971. - Vol. D3. - № 4. - P. 867–873.
2. Aichelburg P. C., Mansouri R., Urbantke H. K. Exact Wave-Type Solution to f-g Theory of Gravity // Phys. Rev. Lett. - 1971. - Vol. 27. - P. 1533–1534.
3. Aichelburg P. C. Implications of classical two-tensor gravity // Phys. Rev. - 1973. - Vol. D8. - № 2. - P. 377–384.
4. Kogan I. I., Ross G. G. Brane universe and multigravity: Modification of gravity at large and small distances // Phys. Lett. - 2000. - Vol. B485. - P. 255–262.
5. Kogan I. I., Mouslopoulos S., Papazoglou A., Ross G. G. Multi-brane worlds and modification of gravity at large scales // Nucl. Phys. - 2001. - Vol. B595. - P. 225–249.
6. Kogan I. I., Mouslopoulos S., Papazoglou A. A new bigravity model with exclusively positive branes // Phys. Lett. - 2001. - Vol. B501. - P. 140–149.
7. Deffayet C., Mourad J. Multigravity from a discrete extra dimension // Phys. Lett. - 2004. - Vol. B589. - P. 48–58.
8. Deffayet C., Mourad J. Some properties of multigravity theories and discretized brane worlds // Int. J. Theor. Phys. - 2004. - Vol. 43. - P. 855–864.
9. Garattini R. Multigravity and space time foam // J. Phys. - 2007. - Vol. A40. - P. 7055–7060.
10. Blas D. Bigravity and massive gravity // AIP Conf. Proc. - 2006. - Vol. 841. - P. 397–401.
11. Hannestad S. Dark energy and dark matter from cosmological observations // Int. J. Mod. Phys. - 2006. - Vol. A21. - P. 1938–1949.
12. Grib A. A., Pavlov Y. V. Superheavy particles and the dark matter problem // Grav. Cosmol. - 2006. - Vol. 12. - P. 159–162.
13. Dubovsky S. L., Tinyakov P. G., Tkachev I. I. Massive graviton as a testable Cold-Dark-Matter candidate // Phys. Rev. Lett. - 2005. - Vol. 94. - № 18. - P. 181102.
14. Damour T., Kogan I. I., Papazoglou A. Non-linear bigravity and cosmic acceleration // Phys. Rev. - 2002. - Vol. D66. - P. 104025.
15. Deffayet C., Dvali G., Gabadadze G. Accelerated universe from gravity leaking to extra dimensions // Phys. Rev. - 2002. - Vol. D65. - № 4. - P. 044023.
16. Damour T., Kogan I. I. Effective lagrangians and universality classes of nonlinear bigravity // Phys. Rev. - 2002. - Vol. D66. - P. 104024.
17. Duplij S. A., Kotvytskiy A. T. Lagrangians in nonlinear multigravity models // J. Kharkov National Univ., ser. Nuclei, Particles and Fields. - 2004. - Vol. 642. - № 3(25). - P. 123–127.
18. Fierz M., Pauli W. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field // Proc. Roy. Soc. Lond. - 1939. - Vol. A173. - P. 211–232.
19. Boulanger N., Damour T., Gualtieri L., Henneaux M. Inconsistency of interacting, multigraviton theories // Nucl. Phys. - 2001. - Vol. B597. - P. 127–171.
20. Landau L. D., Lifshitz E. M. Classical theory of fields. - Oxford: Pergamon Press, 1988. - 376 p.
21. Kogan I. I., Mouslopoulos S., Papazoglou A., Ross G. G., Santiago J. A three three-brane universe: New phenomenology for the new millennium? // Nucl. Phys. - 2000. - Vol. B584. - P. 313–328.

ПРЕДЕЛ СОВПАДЕНИЯ И СТРУКТУРА ОБОБЩЕННОГО ВЗАИМОДЕЙСТВИЯ В МУЛЬТИГРАВИТАЦИИ

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В работе проанализирована обобщенная структура взаимодействия в моделях мультигравитации. В введенном пределе совпадения получено уравнение совместности для потенциала взаимодействия, которое изучается при слабых возмущениях метрики. Исследованы наиболее общие свойства инвариантного объема и скалярного потенциала в мультигравитации. Получена общая формула для инвариантного объема с использованием трех видов средних: арифметического, геометрического и гармонического. В пределе слабого поля для бигравитации получено массовое слагаемое типа Паули-Фирца.

КЛЮЧЕВЫЕ СЛОВА: предел совпадения, уравнение совместности, инвариантный объем, предел слабого поля, скалярный потенциал